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THEORY OF WING OF CIRCULAR PLAN FORM*

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A theory is developed for a wing of circular plan form. The distribution of the bound vortices along the surface of the wing is considered in this theory, which has already been applied in a number of papers. In particular, the case of the circular wing has been examined by Kinner in reference 1.

A second method is considered herein which permits obtaining an expression in closed form for the general solution of this problem. The wing is assumed infinitely thin and slightly cambered and the problem is linearized in the usual manner.

Comparison of the results of the proposed theory with the results of the usual theory of a wing of finite span shows large divergences, which indicate the inadequacy of the usual theory of the case under consideration. For the wings generally employed in practice, which have a considerably greater aspect ratio, a more favorable relation should be obtained between the results of the usual and the more accurate theory.

1. Statement of the Problem

The forward rectilinear motion of a circular wing with constant velocity c is considered. A right-hand system of rectangular coordinates $Oxyz$ is used and the x-axis is taken in the direction of motion of the wing. The wing is assumed thin with a slight camber and has as its projection on the xy-plane a circle ABCD of radius a with center at the origin of the coordinates (fig. 2, in which a section of the wing in the xz-plane is also shown).

Let

$$z = \zeta(x,y) \tag{1.1}$$

represent the equation of the surface of the wing, where the ratio ζ/a as well as the derivatives $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ are assumed to be small magnitudes.

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*"Teoriya kryla konechnogo razmakha krugovoi formy v plane." Prikladnaya Matematika i Mekhanika, Vol. IV. No. 1, 1940, pp. 3-32.

The coordinate axes are assumed to be immovably attached to the wing. The fluid is considered incompressible and the motion nonvortical, steady, and with no acting external forces. The velocity potential of the absolute motion of the fluid will be denoted by $\varphi(x,y,z)$ so that the projection of the absolute velocity of a particle of the fluid is determined by the formulas

$$v_x = \frac{\partial \varphi}{\partial x};\ v_y = \frac{\partial \varphi}{\partial y};\ v_z = \frac{\partial \varphi}{\partial z} \tag{1.2}$$

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The equation of continuity

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

shows that the function φ must satisfy the Laplace equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{1.3}$$

At the leading edge of the wing the velocity of the fluid particles is assumed to become infinite to the order of $1/\sqrt{\delta}$ where δ is the distance of the particle to the leading edge; at the trailing edge the velocity is assumed finite. From the trailing edge of the wing a surface of discontinuity is passed off on which the function φ suffers a discontinuity. The function $\varphi(x,y,z)$ and all its derivatives over the entire space bounded by the said surface of discontinuity and the surface of the wing are continuous.

The problem is linearized in the following manner. The function φ is assumed to suffer a discontinuity on an infinite half-strip Σ located in the xy-plane in the direction of the negative x-axis from the rear semicircumference BCD of the circle S to infinity. In the same manner, the condition on the surface of the wing is replaced by the condition on the surface of the circle S located in the xy-plane and in this way the function $\varphi(x,y,z)$ is assumed to be regular in the region obtained by cutting the infinite half-strip Σ and the circle S from the entire infinite space.

The boundary condition must be satisfied on the surface of the wing.

$$\frac{\partial \varphi}{\partial n} = c \cos(n,x) \tag{1.4}$$

where n is the direction of the normal to the surface of the wing. The direction of this normal, because of the assumption of small curvature of the wing, differs little from the direction of the z-axis. If small terms of the second order are rejected according to the formula

$$\cos(n,x) = -\frac{\frac{\partial\zeta}{\partial x}}{\sqrt{1+\left(\frac{\partial\zeta}{\partial x}\right)^2+\left(\frac{\partial\zeta}{\partial y}\right)^2}} \tag{1.5}$$

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in place of equation (1.4),

$$\frac{\partial\varphi}{\partial z} = -c\frac{\partial\zeta}{\partial x}$$

This condition must be satisfied on the surface of the wing, but it is assumed satisfied on the surface of the circle S, that is, for $z = 0$; this again reduces to the rejection of small terms of the second order by comparison with those of the first order.

The boundary condition is obtained:

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = -c\frac{\partial\zeta(x,y)}{\partial x} \text{ for } x^2+y^2<a^2 \tag{1.6}$$

which must be satisfied on both the upper and lower sides of the circle S.

The boundary conditions are set up which must be satisfied on the surface of discontinuity Σ. On the surface of discontinuity at the trailing edge of the wing, the kinematic condition expresses the continuity of the normal component of the velocity, that is, the magnitude $\partial\varphi/\partial n$ must remain continuous in passing through the surface of discontinuity. Since on the surface of discontinuity the direction of the normal differs little from the direction of the z-axis, transfer of the condition on the surface of discontinuity to the half-strip Σ, gives the equation

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=+0} \left(\frac{\partial\varphi}{\partial z}\right)_{z=-0} \quad \text{for} \quad |y|<a; \quad x^2+y^2>a^2; \quad x<0 \tag{1.7}$$

which expresses the continuity of $\partial\varphi/\partial z$ in passing through the surface of discontinuity Σ.

The dynamical condition expressing the continuity of the pressure in passing through the surface of discontinuity Σ is considered.

In order to determine the pressure p, the formula of Bernoulli is applied to the steady flow about a wing obtained by superposing on the flow considered, a uniform flow with velocity c in the direction of the negative x-axis. In this steady flow the velocity projections are determined by the equations

$$v_x = -c + \frac{\partial\varphi}{\partial x}; \quad v_y = \frac{\partial\varphi}{\partial y}; \quad v_z = \frac{\partial\varphi}{\partial z}$$

and therefore the formula of Bernoulli reduces to the form

$$p = -\frac{\rho}{2}\left[\left(-c + \frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] + \text{constant} \qquad (1.8)$$

Rejection of small terms of the second order results in

$$p = p_0 + \rho c \frac{\partial\varphi}{\partial x} \qquad (1.9)$$

where p_0 is the value of the pressure at infinity.

Since the pressure must remain continuous in passing through the surface of discontinuity at the trailing edge of the wing, the equation obtained shows that $\partial\varphi/\partial x$ does not suffer a discontinuity on the surface of discontinuity. Transfer of this condition to the surface Σ yields the condition

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=+0} = \left(\frac{\partial\varphi}{\partial x}\right)_{z=-0} \quad \text{for} \quad |y| < a; \quad x^2 + y^2 > a^2; \quad x < 0 \qquad (1.10)$$

which expresses the continuity of $\partial\varphi/\partial x$ in passing through Σ.

The function φ suffers a discontinuity on the surfaces S and Σ, which means that along the surfaces S and Σ, surface vortices are located as shown in figure 2. The direction of such a surface vortex is perpendicular to the direction of the relative velocity vector of two particles of the fluid adjacent to the surface of discontinuity on its two sides. In particular, on the surface Σ, on account of equation (1.10), only $\partial\varphi/\partial y$ suffers a discontinuity and therefore the vortex lines on Σ are directed parallel to the x-axis as shown in figure 2.

Since all the vortices lie in the xy-plane, at two points symmetrical with respect to the xy-plane, the values of $\partial\varphi/\partial z$ will be the same, whereas the values of $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ will differ only in sign.

It may therefore be assumed that

$$\varphi(x,y,-z) = -\varphi(x,y,z) \tag{1.11}$$

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Assuming in particular $z = 0$ yields

$$\varphi(x,y,0) = 0$$

in the entire xy-plane with the exception of the circle S and the strip Σ (on which φ suffers a discontinuity).

Since on the strip Σ both condition (1.10) and the condition derived from equation (1.11) must be satisfied

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=+0} = -\left(\frac{\partial\varphi}{\partial x}\right)_{z=-0}$$

and

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=+0} = \left(\frac{\partial\varphi}{\partial x}\right)_{z=-0} = 0 \quad \text{for} \quad |y| < a; \quad x^2 + y^2 > a^2; \quad x < 0 \tag{1.12}$$

Finally, since the fluid far ahead of the wing is assumed to be undisturbed, the condition at infinity is

$$\lim_{x\to+\infty} \frac{\partial\varphi}{\partial x} = \lim_{x\to+\infty} \frac{\partial\varphi}{\partial y} = \lim_{x\to+\infty} \frac{\partial\varphi}{\partial z} = 0 \tag{1.13}$$

In the hydrodynamic problem under consideration, account is taken of the distribution of the vortices along the surface of the wing. It is this circumstance which makes the treatment more accurate than the usual wing theory.

The hydrodynamic problem is thus reduced to the following mathematical problem: To find a harmonic function $\varphi(x,y,z)$ regular over the entire half-space $z > 0$, which on the circle S satisfies the condition

$$\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} = -c\,\frac{\partial\zeta}{\partial x} \tag{1.14}$$

on the strip Σ, the condition

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=0} = 0 \tag{1.15}$$

on the entire remaining part of the xy-plane, the condition

$$\varphi(x,y,0) = 0 \tag{1.16}$$

and the derivatives of which remain bounded in the neighborhood of the rear semicircumference BCD, while in the neighborhood of the forward semicircumference BAD they may approach infinity as $1/\sqrt{\delta}$ where δ is the distance of a point to the semicircumference BAD. Finally the condition at infinity (1.13) must be satisfied.

An expression for the harmonic function $\varphi(x,y,z)$ is given in closed form depending on an arbitrary function $f(x,y)$ satisfying all the imposed requirements besides equation (1.14). The function $\zeta(x,y)$ can be determined from this condition, that is, the shape of the wing corresponding to the function $f(x,y)$. An integral equation is also given, the solution of which is reduced to the determination of the function $f(x,y)$ for the given shape of the wing, that is, for a given function $\zeta(x,y)$.

2. Derivation of the Fundamental Equation

Inside the circle ABCD, the point Q with coordinates ξ, η is taken and the function $K(x,y,z,\xi,\eta)$ constructed, where x,y,z are the coordinates of the point P, according to the following conditions:

(1) The function K, considered as a function of the point P, is a harmonic function outside the circle ABCD.

(2) The function K becomes zero at the points of the plane xy lying outside the circle ABCD.

(3) The derivative $\partial K/\partial z$ becomes zero at all points of the circle ABCD, except the point Q.

(4) When the point P approaches the point Q, remaining in the upper half-space $z > 0$, the function K increases to infinity but the difference $K - (1/r)$, where

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2}$$

remains bounded.

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(5) The function K remains finite and continuous in the neighborhood of the contour C of the circle ABCD.

Because of the second condition, the values of the function K at two points situated symmetrically with respect to the plane xy differ only in sign:

$$K(x,y,-z,\xi,\eta) = -K(x,y,z,\xi,\eta) \tag{2.1}$$

as follows from the principle of analytic continuation. It is then evident that if the third condition is satisfied on the upper side of the circle ABCD it will be satisfied also on the lower side, since according to equation (2.1) the derivative $\partial K/\partial z$ has the same value at two points situated symmetrically with respect to the xy-plane.

It is evident further that when the point P approaches the point Q from below so that $z < 0$ then $K(x,y,z,\xi,\eta)$ will behave as $-1/r$.

Because of the third condition, the function K can be continued into the lower half-space through the upper side of the circle ABCD as an even function of z. Thus a second branch of the function K is assumed, again determined over all the space outside the circle ABCD and differing only in sign from the initial branch of the function K. It is then evident, however, that at the points of the upper side of the circle ABCD, the values of the second branch of the function K and its derivatives coincide with the values of the first branch of the function K and its derivatives at the points of the lower side of the circle ABCD. That is, in the analytic continuation of the second branch of the function K through the upper side of the circle ABCD into the lower half-space, the initial branch of this function is again obtained.

A two-sheet Riemann space is considered for which the branching line is the circumference ABCD. In this space $K(x,y,z,\xi,\eta)$ is a single-valued harmonic function remaining finite everywhere with the exception of the two points Q having the same coordinates $(\xi,\eta,0)$, but belonging to two different sheets of space; in one sheet the function K behaves near the point Q as $1/r$ and in the other sheet as $-1/r$. Such a function $K(x,y,z,\xi,\eta)$ can readily be constructed by the method of Sommerfeld (reference 2). In this way for the case of a two-sheet Riemann space having as branch line the z-axis, a harmonic function $V(\rho,\varphi,z)$ (ρ,φ,z being the cylindrical coordinates of the point) is determined which is single-valued and continuous in the entire two-sheet space with the exception of the points Q and Q' having the cylindrical coordinates (ρ',φ',z') and $(\rho',-\varphi',z')$, where near the point Q the function V behaves as $1/r$ and near the point Q' as $-1/r$, where

$$r = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi - \varphi') + (z - z')^2}$$

$$r' = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\varphi + \varphi') + (z - z')^2}$$

This function V has the form:

$$V = \frac{2}{\pi}\left\{\frac{1}{r}\text{arc tan}\sqrt{\frac{\sigma + \tau}{\sigma - \tau}} - \frac{1}{r'}\text{arc tan}\sqrt{\frac{\sigma + \tau'}{\sigma - \tau'}}\right\}$$

where

$$\sigma = \frac{1}{2\sqrt{\rho\rho'}}\sqrt{(\rho + \rho')^2 + (z - z')^2};\quad \tau = \cos\frac{\varphi - \varphi'}{2};\quad \tau' = \cos\frac{\varphi + \varphi'}{2}$$

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Setting, in particular,

$$\varphi' = \pi;\quad r = \sqrt{\rho^2 + \rho'^2 + 2\rho\rho'\cos\varphi + (z - z')^2}$$

yields

$$V = \frac{2}{\pi r}\left\{\text{arc tan}\sqrt{\frac{\sigma + \tau}{\sigma - \tau}} - \text{arc tan}\sqrt{\frac{\sigma - \tau}{\sigma + \tau}}\right\} = \frac{2}{\pi r}\text{arc tan}\frac{\tau}{\sqrt{\sigma^2 - \tau^2}}$$

or finally

$$V = \frac{2}{\pi r}\text{arc tan}\frac{2\sqrt{\rho\rho'}\sin\frac{\varphi}{2}}{r}$$

An inversion with respect to the point with coordinates $\rho = a$, $\varphi = 0$, $z = 0$ is carried out.

$$\rho\cos\varphi = a + \frac{2a^2(x_1 - a)}{(x_1 - a)^2 + y_1^2 + z_1^2};\quad -\rho' = a + \frac{2a^2(\xi_1 - a)}{(\xi_1 - a)^2 + \zeta_1^2}$$

$$\rho\sin\varphi = \frac{2a^2 y_1}{(x_1 - a)^2 + y_1^2 + z_1^2};\quad z = \frac{2a^2 z_1}{(x_1 - a)^2 + y_1^2 + z_1^2}$$

$$z' = \frac{2a^2\zeta_1}{(\xi_1 - a)^2 + \zeta_1^2}$$

The function

$$V_1 = \frac{2a^2 V}{\sqrt{(x_1 - a)^2 + y_1^2 + z_1^2}\sqrt{(\xi_1 - a)^2 + \zeta_1^2}}$$

expressed in the variables x_1, y_1, z_1 is then, as is known, a harmonic function. Computing it and replacing x_1, y_1, z_1 by y, z, x and ξ_1, ζ_1 by η, ξ yield the required expression of the function $K(x,y,z,\xi,\eta)$:

$$K(x,y,z,\xi,\eta) = \frac{2}{\pi r} \operatorname{arc} \tan \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\, ar} \tag{2.2}$$

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valid for $z > 0$, where

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2}$$
$$R = \sqrt{(a^2 - x^2 - y^2 - z^2)^2 + 4a^2z^2} = \sqrt{(a^2 + x^2 + y^2 + z^2)^2 - 4a^2(x^2 + y^2)} \tag{2.3}$$

That this function satisfies all the above set requirements is easily verified; the arc tangents must be taken between 0 and $\pi/2$; for $z < 0$ the value of the function K is obtained by equation (2.1).

The following function is set up:

$$\varphi_1(x,y,z) = \frac{1}{2\pi} \iint_S K(x,y,z,\xi,\eta) f(\xi,\eta) d\xi d\eta \tag{2.4}$$

where $f(x,y)$ is an arbitrary function, which is continuous together with its partial derivatives of the first and second order in the entire circle S, and the integration extends over the entire area of the circle S. Evidently, $\varphi_1(x,y,z)$ is a harmonic function in the entire space outside the circle S. Because of the first property of the function K, the function $\varphi_1(x,y,z)$ becomes zero at all points of the plane xy which are outside the circle S. Hence equations (1.15) and (1.16), which must be satisfied by the solution $\varphi(x,y,z)$ of the problem posed in section 1, will be satisfied for the function $\varphi_1(x,y,z)$. The function $\varphi_1(x,y,z)$ does not in general satisfy the condition of the finiteness of the derivatives of this function on the rear half of the contour of the circle S. For this reason, a function such that the obtained function $\varphi(x,y,z)$ also satisfies this condition is added to $\varphi_1(x,y,z)$.

The following equation is evident:

$$\frac{\partial \varphi_1}{\partial x} = \frac{1}{2\pi} \iint_S \frac{\partial K}{\partial x} f(\xi,\eta) d\xi d\eta$$

The character of the approach of the function $\partial K/\partial x$ to infinity is considered as a point approaches the contour C of the circle S. As may be easily computed

$$\frac{\partial K}{\partial x} = -\frac{2(x-\xi)}{\pi r^3}\arc\tan\frac{\sqrt{a^2-\xi^2-\eta^2}\sqrt{a^2-x^2-y^2-z^2+R}}{\sqrt{2}\,ar} -$$

$$\frac{2\sqrt{2}a}{\pi}\frac{\sqrt{a^2-\xi^2-\eta^2}\sqrt{a^2-x^2-y^2-z^2+R}}{2a^2r^2+(a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)}\left\{\frac{x-\xi}{r^2}+\frac{x}{R}\right\} \tag{2.5}$$

If a point with coordinates x,y,z is near the contour C of the circle S the distance of this point to the contour C is denoted by δ; then

$$\delta = \sqrt{a^2+x^2+y^2+z^2-2a\sqrt{x^2+y^2}} \tag{2.6}$$

Hence near the contour C, the approximate equation holds:

$$R = 2a\delta \tag{2.7}$$

When the fixed point ξ,η lies inside the circle S while the point with coordinates x,y,z lies near the contour C of the circle, then, as follows from equation (2.5),

$$\frac{\partial K}{\partial x} = -\frac{\sqrt{2}\,x}{\pi ar^2R}\sqrt{a^2-\xi^2-\eta^2}\sqrt{a^2-x^2-y^2-z^2+R}+O(1) \tag{2.8}$$

where the symbol $O(1)$ denotes a magnitude which remains finite when δ approaches zero. Thus $\partial K/\partial x$ has the order $1/\sqrt{\delta}$. The principal part of $\partial K/\partial x$ is not a harmonic function. It is not difficult, however, to find a harmonic function having the same infinite part near the contour C as $\partial K/\partial x$. For this, it is sufficient to form, after the analogy of equation (2.5), the derivative $\partial K/\partial\xi$; this derivative remains finite near the contour C of the wing; moreover it is easy to see that

$$\frac{\partial K}{\partial x}+\frac{\partial K}{\partial\xi} = -\frac{2\sqrt{2}\,a\sqrt{a^2-x^2-y^2-z^2+R}}{\pi\left[2a^2r^2+(a^2-\xi^2-\eta^2)(a^2-x^2-y^2-z^2+R)\right]}\cdot$$

$$\left\{\frac{x\sqrt{a^2-\xi^2-\eta^2}}{R}+\frac{\xi}{\sqrt{a^2-\xi^2-\eta^2}}\right\} \tag{2.9}$$

This function is harmonic and differs from $\partial K/\partial x$ by a quantity which remains finite near the contour C.

By computation, it is further shown that the function just described is represented in the form of the integral

$$\frac{\partial K}{\partial x} + \frac{\partial K}{\partial \xi} = -\frac{1}{\pi^2\sqrt{2}} \int_{-\pi}^{\pi} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma\, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{2.10}$$

where the function

$$\frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma}$$

is a solution of the equation of Laplace having the circumference C as the branching line and the point with coordinates $(a\cos\gamma,\ a\sin\gamma,\ 0)$ as a singular point. From this it follows that the function

$$\frac{\partial K}{\partial x} + \frac{1}{\pi^2\sqrt{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma\, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}$$

$$= -\frac{\partial K}{\partial \xi} - \frac{1}{\pi^2\sqrt{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma\, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{2.11}$$

remains finite near the points of the rear semicircumference of the circle S.

Therefore it is assumed

$$\frac{\partial \varphi}{\partial x} = \frac{1}{2\pi}\iint_S f(\xi,\eta)\left\{\frac{\partial K}{\partial x} + \frac{1}{\pi^2\sqrt{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma\, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)}\right\} d\xi\, d\eta \tag{2.12}$$

Integrating with respect to x and considering the condition at infinity (1.13) yield the final equation

$$\varphi(x,y,z) = \frac{1}{2\pi} \iint_S f(\xi,\eta) \Bigg\{ K(x,y,z,\xi,\eta) + \frac{1}{\pi^2\sqrt{2}} \int_{+\infty}^{x} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma \, d\gamma \, dx}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \Bigg\} d\xi \, d\eta \tag{2.13}$$

This equation may be written in somewhat different form. Because of equation (2.11)

$$\frac{\partial\varphi}{\partial x} = -\frac{1}{2\pi} \iint_S \frac{\partial K}{\partial \xi} f(\xi,\eta) d\xi \, d\eta - \frac{1}{2\sqrt{2}\,\pi^3} \times$$

$$\iint_S \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}\cos\gamma \, d\gamma}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} f(\xi,\eta) d\xi \, d\eta$$

Since the function K becomes zero on the contour C

$$\iint_S \frac{\partial K}{\partial \xi} f(\xi,\eta) d\xi \, d\eta = -\iint_S K \frac{\partial f}{\partial \xi} d\xi \, d\eta \tag{2.14}$$

Introduction of further notations

$$-\frac{1}{2\pi^3\sqrt{2}} \iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta) d\xi \, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} = G(\gamma); \quad \frac{\partial f}{\partial \xi} = F(\xi,\eta) \tag{2.15}$$

results in

$$\frac{\partial\varphi}{\partial x} = \frac{1}{2\pi} \iint_S K(x,y,z,\xi,\eta) F(\xi,\eta) d\xi \, d\eta +$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}\; G(\gamma)\cos\gamma}{x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma} d\gamma \tag{2.16}$$

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and after integration with respect to x

$$\varphi(x,y,z) = \frac{1}{2\pi}\iint_S\int_{+\infty}^{x} K(x,y,z,\xi,\eta)dx\, F(\xi,\eta)d\xi\, d\eta +$$

$$\int_{+\infty}^{x}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}\, G(\gamma)\cos\gamma}{x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma} dx\, d\gamma \qquad (2.17)$$

The given functions $F(\xi,\eta)$ in the circle S and the function $G(\gamma)$ in the interval $(-\pi/2, \pi/2)$ completely determine $f(\xi,\eta)$, so that the equations (2.13) and (2.17) are equivalent.

The equation $\varphi(x,y,z)$ obtained satisfies the conditions imposed in section 1.

This function is evidently a harmonic function in the entire space exterior to the circle S and satisfies the conditions at infinity, equation (1.13). From equation (2.12) it follows, that in the plane xy for $x^2 + y^2 > a^2$ the condition is satisfied:

$$\left(\frac{\partial\varphi}{\partial x}\right)_{z=0} = 0$$

and from equation (2.13) it follows that

$$\varphi(x,y,0) = 0$$

in that part of the plane xy which lies outside the circle S and the strip Σ.

It remains to prove the finiteness of the first derivatives of the function $\varphi(x,y,z)$ at the points of the rear semicircumference C and to determine the behavior of these derivatives on approaching the points of the forward semicircumference C.

In considering the neighborhood of the rear side of the circumference C, equation (2.16) may be used. The latter shows that $\partial\varphi/\partial x$ remains continuous at the points of the rear half of the circumference C and becomes zero at these points.

The behavior of the derivatives with respect to y and z of the following function is considered:

$$\Phi(x,y,z) = \iint_S K(x,y,z,\xi,\eta)F(\xi,\eta)d\xi\, d\eta \qquad (2.18)$$

near the contour C.

$$\frac{\partial\Phi}{\partial y} = \iint_S \frac{\partial K}{\partial y} F(\xi,\eta)d\xi\, d\eta \qquad (2.19)$$

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Similarly to equation (2.9),

$$\frac{\partial K}{\partial y} + \frac{\partial K}{\partial \eta} = -\frac{2a\sqrt{2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\pi\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]}$$

$$\left\{\frac{y\sqrt{a^2 - \xi^2 - \eta^2}}{R} + \frac{\eta}{\sqrt{a^2 - \xi^2 - \eta^2}}\right\} \qquad (2.20)$$

and similarly to equation (2.14),

$$\iint_S \frac{\partial K}{\partial \eta} F(\xi,\eta)d\xi\, d\eta = -\iint_S K \frac{\partial F}{\partial \eta} d\xi\, d\eta \qquad (2.21)$$

where this part of the integral remains finite everywhere and on the contour C becomes zero.

In order to evaluate the remaining part of the integral equation (2.19), the following two integrals are considered:

$$\left.\begin{aligned} J_1(x,y,z) &= \iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)} \\ J_2(x,y,z) &= \iint_S \frac{d\xi\, d\eta}{\sqrt{a^2 - \xi^2 - \eta^2}\left[2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)\right]} \end{aligned}\right\} \qquad (2.22)$$

Both, on account of the symmetry, depend only on $\sqrt{x^2 + y^2}$ and z; hence without restricting the generality, it may be assumed that $y = 0$, $x > 0$. The distance δ of a point with coordinates $(x,0,z)$ is introduced to the contour C:

$$\delta = \sqrt{(a - x)^2 + z^2}$$

Since

$$R \geq |x^2 + z^2 - a^2|$$

the following relation will hold:

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$$J_1(x,0,z) \leq \iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2\left[(x - \xi)^2 + \eta^2 + z^2\right]}$$

Polar coordinates are introduced

$$\xi = \rho \cos \vartheta \; ; \; \eta = \rho \sin \vartheta$$

whence

$$J_1(x,0,z) \leq \int_0^a \int_0^{2\pi} \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho\, d\vartheta}{2a^2\left[\rho^2 - 2\rho x \cos \vartheta + x^2 + z^2\right]}$$

Since

$$\int_0^{2\pi} \frac{d\vartheta}{\rho^2 - 2\rho x \cos \vartheta + x^2 + z^2} = \frac{2\pi}{\sqrt{(\rho^2 + x^2 + z^2)^2 - 4\rho^2 x^2}}$$

hence

$$J_1(x,0,z) \leq \frac{\pi}{a^2} \int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{\sqrt{(\rho^2 + x^2 + z^2)^2 - 4\rho^2 x^2}}$$

For $x \geq a$

$$J_1(x,0,z) \leq \frac{\pi}{a^2} \int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{\sqrt{(\rho^2 + x^2)^2 - 4\rho^2 x^2}}$$

$$= \frac{\pi}{a^2} \int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{x^2 - \rho^2} \leq \frac{\pi}{a^2} \int_0^a \frac{\rho d\rho}{\sqrt{a^2 - \rho^2}} = \frac{\pi}{a}$$

While for $x \leqslant a$, use is made of the inequality

$$R \geqslant a^2 - x^2 - z^2$$

to obtain

$$J_1(x,0,z) \leqslant \frac{1}{2}\iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{a^2\left[(x - \xi)^2 + \eta^2 + z^2\right] + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - z^2)}$$ 2439

$$= \frac{1}{2}\int_0^a \int_0^{2\pi} \frac{\rho\sqrt{a^2 - \rho^2}\, d\vartheta\, d\rho}{a^4 - 2a^2 x\rho \cos\vartheta + \rho^2(x^2 + z^2)} \tag{2.23}$$

$$= \pi\int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{\sqrt{\left[a^4 + \rho^2(x^2 + z^2)\right]^2 - 4a^4x^2\rho^2}} \leqslant \pi\int_0^a \frac{\rho\sqrt{a^2 - \rho^2}\, d\rho}{a^4 - x^2\rho^2} \leqslant \pi\int_0^a \frac{\rho\, d\rho}{a^2\sqrt{a^2 - \rho^2}} = \frac{\pi}{a}$$

The following inequality results:

$$J_1(x,y,z) \leqslant \frac{\pi}{a} \tag{2.24}$$

The second integral is considered. As before,

$$J_2(x,0,z) \leqslant \frac{\pi}{a^2}\int_0^a \frac{\rho\, d\rho}{\sqrt{a^2 - \rho^2}\sqrt{(\rho^2 + x^2 + z^2)^2 - 4\rho^2x^2}}$$

For $x \geqslant a$

$$J_2(x,0,z) \leqslant \frac{\pi}{a^2}\int_0^a \frac{\rho\, d\rho}{\sqrt{(a^2 - \rho^2)\left[(\rho + x)^2 + z^2\right]\left[(\rho - x)^2 + z^2\right]}} \leqslant \frac{\pi}{a^3}\int_0^a \frac{\rho\, d\rho}{\sqrt{a^2 - \rho^2}\sqrt{(a - x)^2 + z^2}} = \frac{\pi}{a^2\delta}$$

For $x \leqslant a$ an inequality of the type in equation (2.23) is used:

$$J_2(x,0,z) \leqslant \pi\int_0^a \frac{\rho\, d\rho}{\sqrt{(a^2 - \rho^2)\left[a^4 + \rho^2(x^2 + z^2) + 2a^2x\rho\right]\left[(a^2 - x\rho)^2 + \rho^2z^2\right]}}$$

$$\leqslant \frac{\pi}{a^2}\int_0^a \frac{\rho\, d\rho}{\sqrt{(a^2 - \rho^2)\left[(a^2 - x\rho)^2 + z^2\rho^2\right]}}$$

If $z \geq a - x$ and therefore $\delta \leq z\sqrt{2}$, then

$$J_2(x,0,z) \leq \frac{\pi}{a^2 z}\int_0^a \frac{d\rho}{\sqrt{a^2 - \rho^2}} = \frac{\pi^2}{2a^2 z} \leq \frac{\pi^2}{a^2\delta\sqrt{2}}$$

but if $0 \leq z \leq a - x$, and therefore $\delta \leq (a - x)\sqrt{2}$, then

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$$J_2(x,0,z) \leq \frac{\pi}{a^2}\int_0^a \frac{\rho\, d\rho}{(a^2 - x\rho)\sqrt{a^2 - \rho^2}} < \frac{\pi}{a^3}\int_0^a \frac{\rho\, d\rho}{(a - x)\sqrt{a^2 - \rho^2}} = \frac{\pi}{a^2(a - x)} \leq \frac{\pi\sqrt{2}}{a^2\delta}$$

The following approximation is obtained:

$$J_2(x,y,z) \leq \frac{\pi^2}{a^2\delta\sqrt{2}} \tag{2.25}$$

where

$$\delta = \sqrt{\left(a - \sqrt{x^2 + y^2}\right)^2 + z^2} \tag{2.26}$$

Near the contour C

$$R = 2a\delta \tag{2.27}$$

If this relation, the evident inequality

$$|a^2 - x^2 - y^2 - z^2| \leq R$$

and the obtained approximations are used, the following approximation is obtained from equation (2.20):

$$\left|\iint_S \left(\frac{\partial K}{\partial y} + \frac{\partial K}{\partial \eta}\right) F(\xi,\eta)\, d\xi\, d\eta\right| = 0\left(\frac{1}{\sqrt{\delta}}\right)$$

It is evident from equations (2.19) and (2.21) that near the contour C

$$\frac{\partial \Phi}{\partial y} = 0\left(\frac{1}{\sqrt{\delta}}\right) \tag{2.28}$$

The following derivative is formed:

$$\frac{\partial \Phi}{\partial z} = \iint_S \frac{\partial K}{\partial z} F(\xi,\eta) d\xi\, d\eta$$

But

$$\frac{\partial K}{\partial z} = -\frac{2z}{\pi r^3} \operatorname{arc\,tan} A + \frac{2}{\pi} \frac{A}{1 + A^2} \left[-\frac{z}{r^3} + \frac{z(a^2 + x^2 + y^2 + z^2 - R)}{rR(a^2 - x^2 - y^2 - z^2 + R)} \right]$$ 2439

where

$$A = \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{ar\sqrt{2}}$$

Hence if

$$|F(\xi,\eta)| < M$$

then, on account of the inequality

$$\frac{A}{1 + A^2} \leqslant \frac{1}{2}$$

for $z > 0$ the approximation results:

$$\left|\frac{\partial \Phi}{\partial z}\right| \leqslant 2M \iint_S \frac{z}{r^3} d\xi\, d\eta +$$

$$\frac{2\sqrt{2}\, azM(a^2 + x^2 + y^2 + z^2 - R)}{\pi R\sqrt{a^2 - x^2 - y^2 - z^2 + R}} \iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{2a^2r^2 + (a^2 - \xi^2 - \eta^2)(a^2 - x^2 - y^2 - z^2 + R)}$$

Noting that

$$\iint_S \frac{z}{r^3} d\xi\, d\eta \leqslant 2\pi$$

and making use of approximation (2.24) yield

$$\left|\frac{\partial\Phi}{\partial z}\right| \leqslant 4\pi M + 2\sqrt{2}\, M \frac{z(a^2 + x^2 + y^2 + z^2 - R)}{R\sqrt{a^2 - x^2 - y^2 - z^2 + R}}$$

Since for $z > 0$

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$$\frac{z}{\sqrt{a^2 - x^2 - y^2 - z^2 + R}} = \frac{z\sqrt{R - (a^2 - x^2 - y^2 - z^2)}}{\sqrt{R^2 - (a^2 - x^2 - y^2 - z^2)^2}} = \frac{\sqrt{R - a^2 + x^2 + y^2 + z^2}}{2a}$$

hence

$$\left|\frac{\partial\Phi}{\partial z}\right| \leqslant 4\pi M + \frac{\sqrt{2}\, M}{aR}(a^2 + x^2 + y^2 + z^2 - R)\sqrt{R - a^2 + x^2 + y^2 + z^2}.$$

Now when the point $P(x,y,z)$ is near the contour C, then because of

$$R = 2a\delta \; ; \; |x^2 + y^2 + z^2 - a^2| \leqslant R$$

there is obtained

$$\left|\frac{\partial\Phi}{\partial z}\right| = 0\left(\frac{1}{\sqrt{\delta}}\right) \tag{2.29}$$

Equation (2.16) is again considered. Since the derivatives

$$\left.\begin{aligned} &\frac{\partial}{\partial y}\sqrt{a^2 - x^2 - y^2 - z^2 + R} = -\frac{y\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{R}; \\ &\frac{\partial}{\partial z}\sqrt{a^2 - x^2 - y^2 - z^2 + R} = \frac{z(a^2 + x^2 + y^2 + z^2 - R)}{R\sqrt{a^2 - x^2 - y^2 - z^2 + R}} \\ &= \frac{1}{2aR}(a^2 + x^2 + y^2 + z^2 - R)\sqrt{R - a^2 + x^2 + y^2 + z^2} \end{aligned}\right\} \tag{2.30}$$

have near the contour C the order $1/\sqrt{\delta}$, it is clear from equation (2.16) and the obtained equations (2.28) and (2.29) that at the points of the rear semicircumference of C there is the estimate

$$\frac{\partial^2\varphi}{\partial x\partial y} = 0\left(\frac{1}{\sqrt{\delta}}\right); \quad \frac{\partial^2\varphi}{\partial x\partial z} = 0\left(\frac{1}{\sqrt{\delta}}\right) \tag{2.31}$$

But is is then evident that the derivatives $\partial\varphi/\partial y$ and $\partial\varphi/\partial z$ are finite at the points of the rear semicircumference C.

The behavior of the derivatives of the function φ near the forward semicircumference C can readily be determined, starting from equations (2.12) and (2.13).

The first of these equations may be written in the form:

$$\frac{\partial\varphi}{\partial x} = \frac{1}{2\pi}\iint_S \frac{\partial K}{\partial x} f(\xi,\eta)d\xi\, d\eta - \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}\; G(\gamma) \cos\gamma}{x^2 + y^2 + z^2 + a^2 - 2ax \cos\gamma - 2ay \sin\gamma} d\gamma \tag{2.32}$$

But on the one hand, the estimate

$$\iint_S \frac{\partial K}{\partial x} f(\xi,\eta)d\xi\, d\eta = 0\left(\frac{1}{\sqrt{\delta}}\right)$$

holds for the neighborhood of the entire contour C; on the other hand, on the forward semicircumference C, the second integral of equation (2.32) evidently remains finite. Hence for the forward semicircumference C the first of the estimates is obtained

$$\frac{\partial\varphi}{\partial x} = 0\left(\frac{1}{\sqrt{\delta}}\right); \quad \frac{\partial\varphi}{\partial y} = 0\left(\frac{1}{\sqrt{\delta}}\right); \quad \frac{\partial\varphi}{\partial z} = 0\left(\frac{1}{\sqrt{\delta}}\right) \tag{2.33}$$

while the latter two of these estimates are obtained in a similar manner from equation (2.13).

In this manner all the conditions which must be satisfied by the function $\varphi(x,y,z)$ are satisfied.

The shape of wing to which the obtained solution corresponds is explained. By equation (1.14)

$$c\frac{\partial\zeta}{\partial x} = -\left(\frac{\partial\varphi}{\partial z}\right)_{z=0} \tag{2.34}$$

Hence it is necessary to find the value $\partial\varphi/\partial z$ in the plane of the circle S. Both sides of equation (2.13) are differentiated with respect to z and then z set = 0. On account of the very definition of the function K,

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$$\lim_{z \to +0} \iint_S \frac{\partial K}{\partial z} f(\xi,\eta) d\xi \, d\eta = \lim_{z \to +0} \iint_S \frac{\partial \frac{1}{r}}{\partial z} f(\xi,\eta) d\xi \, d\eta = -2\pi f(x,y) \tag{2.35}$$

Moreover, on account of equation (2.30),

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$$\lim_{z \to +0} \frac{\partial}{\partial z} \sqrt{a^2 - x^2 - y^2 - z^2 + R} = \begin{cases} 0 & \text{for } x^2 + y^2 < a^2 \\ \dfrac{a\sqrt{2}}{\sqrt{x^2 + y^2 - a^2}} & \text{for } x^2 + y^2 > a^2 \end{cases}$$

If this is taken into account,

$$\left(\frac{\partial \varphi}{\partial z}\right)_{z=0} = -f(x,y) + g(y) \tag{2.36}$$

where

$$g(y) = \frac{a}{2\pi^3} \iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2} \cos \gamma f(\xi,\eta) d\gamma \, dx \, d\xi \, d\eta}{\sqrt{x^2 + y^2 - a^2}(x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2a\eta \sin \gamma)} \tag{2.37}$$

For the function $\zeta(x,y)$ the following expression is found:

$$\zeta(x,y) = \frac{1}{c} \int_0^x f(x,y) dx - \frac{g(y)}{c} x + g_1(y) \tag{2.38}$$

where $g_1(y)$ is an arbitrary function of y.

Thus, for the assumed degree of approximation, the bending of the wing in the transverse direction produces no effect on the form of the flow.

It is assumed that the shape of the wing is given, that is, the function $\zeta(x,y)$ and therefore the following function are given:

$$c \frac{\partial \zeta}{\partial x} = M(x,y) \tag{2.39}$$

From equations (2.34) and (2.36) it is clear that

$$f(x,y) = M(x,y) + g(y) \tag{2.40}$$

Substituting this value in equation (2.37) and introducing the notations

$$N(y) = \frac{a}{2\pi^3} \times \iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2-\xi^2-\eta^2}\, M(\xi,\eta) \cos\gamma \, d\gamma \, dx \, d\xi \, d\eta}{\sqrt{x^2+y^2-a^2}\,(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)(\xi^2+\eta^2+a^2-2a\xi\cos\gamma-2a\eta\sin\gamma)}$$

$$H(y,\eta) = \frac{a}{2\pi^3} \times \int_{-\sqrt{a^2-\eta^2}}^{\sqrt{a^2-\eta^2}} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2-\xi^2-\eta^2} \cos\gamma \, d\gamma \, dx \, d\xi}{\sqrt{x^2+y^2-a^2}\,(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma)(\xi^2+\eta^2+a^2-2a\xi\cos\gamma-2a\eta\sin\gamma)} \tag{2.41}$$

give an integral Fredholm equation of the second kind for the determination of the function $g(y)$:

$$g(y) = N(y) + \int_{-a}^{a} H(y,\eta) g(\eta) d\eta \tag{2.42}$$

In consideration of examples, a function $f(x,y)$ shall be given and the shape of the wing then determined by equation (2.38). For the obtained shapes of the wing it is not difficult to find a solution by the usual theory, a fact which provides the possibility of evaluating the degree of accuracy of the usual theory.

3. Computation of the Forces Acting on the Wing

The fundamental equation determining the motion of the type under consideration is recalled:

$$\varphi(x,y,z) = \frac{1}{2\pi} \iint_S \left\{ K(x,y,z,\xi,\eta) + \frac{1}{\pi^2\sqrt{2}} \times \int_{+\infty}^{x} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2-\xi^2-\eta^2}\sqrt{a^2-x^2-y^2-z^2+R}\cos\gamma \, d\gamma \, dx}{(x^2+y^2+z^2+a^2-2ax\cos\gamma-2ay\sin\gamma)(\xi^2+\eta^2+a^2-2a\xi\cos\gamma-2a\eta\sin\gamma)} \right\} f(\xi,\eta) d\xi \, d\eta \tag{3.1}$$

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The value of the function φ for the points of the half-strip Σ is computed. Since at the points of the half-strip Σ

$$\frac{\partial\varphi}{\partial x} = 0$$

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this value is a function only of y. The notation is introduced

$$\Phi(y) = \lim_{z \to +0} \varphi(x,y,z) \quad \text{for} \quad |y| < a, \quad x^2 + y^2 > a^2, \quad x < 0 \tag{3.2}$$

Then evidently

$$\lim_{z \to -0} \varphi(x,y,z) = -\Phi(y) \quad \text{for} \quad |y| < a, \quad x^2 + y^2 > a^2, \quad x < 0 \tag{3.3}$$

The circulation over the contour M'NM (fig. 2) connecting the two points M and M' of which point M' lies on the lower and point M the upper side of the half-strip Σ, both points M and M' having the same coordinates $x,y,0$, is denoted by $\Gamma(y)$. It is then evident that

$$\Gamma(y) = \Phi(M) - \Phi(M') = 2\Phi(y) \tag{3.4}$$

Since in the plane xy outside the circle S both the function K and the function

$$\sqrt{a^2 - x^2 - y^2 - z^2 + R}$$

become zero, it is clear that

$$\Phi(y) = \frac{1}{2\pi^3} \iint_S \int_{\sqrt{a^2-y^2}}^{-\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2}\, f(\xi,\eta) \cos\gamma \, d\gamma \, dx \, d\xi \, d\eta}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{3.5}$$

Computation shows that

$$\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{\sqrt{a^2 - x^2 - y^2}\, dx}{x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma} = \pi\left\{\sqrt{\frac{a(1 \pm \sin \gamma)}{|a \sin \gamma - y|}} - 1\right\} \quad (3.6)$$

where the plus sign is taken for $y < a \sin \gamma$ and the minus sign for $y > a \sin \gamma$.

The following expression is written for the distribution of the circulation in the vortex layer formed behind the wing:

$$\Gamma(y) = -\frac{1}{\pi^2} \iint_S \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)}{(\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2a\eta \sin \gamma)}\left\{\sqrt{\frac{a(1 \pm \sin \gamma)}{|a \sin \gamma - y|}} - 1\right\} \cos \gamma \, d\gamma \, d\xi \, d\eta \quad (3.7)$$

The forces acting on the wing are computed. Denoting by p_+ the pressure at a point of the wing S on the upper side of the wing and by p_- the pressure at the same point on the lower side gives on the basis of equation (1.8)

$$p_- - p_+ = -2\rho c \frac{\partial \varphi}{\partial x} \quad (3.8)$$

where the value of $\partial\varphi/\partial x$ is taken on the upper side of the wing.

For the lift force P, the following expression is obtained:

$$P = \iint_S (p_- - p_+)\, dx\, dy = -2\rho c \iint_S \frac{\partial \varphi}{\partial x}\, dx\, dy = -2\rho c \int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{\partial \varphi}{\partial x}\, dx\, dy$$

$$= -2\rho c \int_{-a}^{a} \left[\varphi(\sqrt{a^2 - y^2}, y, 0) - \varphi(-\sqrt{a^2 - y^2}, y, 0)\right] dy = 2\rho c \int_{-a}^{a} \phi(y)\, dy$$

The following formula is obtained:

$$P = \rho c \int_{-a}^{a} \Gamma(y) dy \quad (3.9)$$

having the same form as in the usual theory of a wing of finite span. But the distribution of the circulation $\Gamma(y)$ by the present theory is somewhat different from that obtained by the usual theory. The derivation given is not connected with the shape of the wing.

With the aid of equation (3.6) P may be directly expressed through $f(\xi,\eta)$:

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$$P = -\frac{2\rho a c}{\pi^2}\iint_S \sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta) \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\, d\xi\, d\eta \tag{3.10}$$

The expression for the induced resistance W in terms of the circulation $\Gamma(y)$ likewise has the same form as in the usual theory:

$$W = \frac{\rho}{4\pi}\int_{-a}^{a}\int_{-a}^{a} \Gamma(y)\frac{d\Gamma(y')}{dy'}\frac{1}{y - y'}\, dy\, dy' \tag{3.11}$$

because the origin of the induced resistance is due to the fact that behind the wing a region of disturbed motion of the fluid is formed; the kinetic energy of this disturbance is determined on the other hand exclusively by the distribution of the circulation at distant points from the wing.

The expression for the induced resistance is obtained from the momentum law.

A surface enclosing the wing S is denoted by B; the momentum law applied to the wing in a steady flow then leads to the expression

$$W = \iint_B p\cos(n,x)d\sigma + \iint_B \rho V_n V_x d\sigma \tag{3.12}$$

where n is the direction of the outer normal to the surface B and V_x, V_y, V_z are the components of the velocity in the relative motion of the fluid about the wing. Thus

$$V_x = -c + \frac{\partial\varphi}{\partial x};\quad V_n = -c\cos(n,x) + \frac{\partial\varphi}{\partial n}$$

$$p = p_0 + \rho c\frac{\partial\varphi}{\partial x} - \frac{\rho}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]$$

Substituting these values in the preceding formula and noting that

$$\iint_B \cos(n,x)\,d\sigma = 0; \quad \iint_B \frac{\partial\varphi}{\partial n}\,d\sigma = 0$$

results in

$$W = -\frac{\rho}{2}\iint_B \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right]\cos(n,x)\,d\sigma + \rho\iint_B \frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial n}\,d\sigma \quad (3.13)$$

The surface B consists of a hemisphere of large radius with center at the point $x = x_0 < -a$ of the x-axis enclosing the wing, and of the circle cut out by this hemisphere on the plane $x = x_0$. With increase in the radius of the hemisphere to infinity the corresponding parts of the integrals entering the preceding formula approach zero. On the surface $x = x_0$

$$\cos(n,x) = -1; \quad \frac{\partial\varphi}{\partial n} = -\frac{\partial\varphi}{\partial x}$$

therefore

$$W = \frac{\rho}{2}\iint \left[\left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2\right]dy\,dz \quad (3.14)$$

where the integration extends over the entire plane $x = x_0$. For $x_0 \to -\infty$ the following equation is obtained:

$$W = \frac{\rho}{2}\iint \left[\left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2\right]dy\,dz \quad (3.15)$$

where $\Phi(y,z)$ denotes the velocity potential of the plane-parallel flow which is established in the transverse planes far behind the wing.

The usual transformations by Green's formula yield

$$W = -\rho\int_{-a}^{a} \Phi(y)\,\frac{\partial\Phi}{\partial z}\,dy \quad (3.16)$$

where the integral is taken over the upper side of the segment $(-a,a)$ in the plane yz.

Since

$$\Gamma(y) = 2\Phi(y); \quad \frac{\partial\Phi}{\partial z} = \frac{1}{2\pi}\int_{-a}^{a} \frac{d\Gamma(y')}{y' - y} \quad (3.17)$$

equation (3.11) is obtained.

In order to find the center of pressure, the principal moments of the pressure forces about the Ox and Oy axes are determined.

For the moment about the Ox axis,

$$M_x = \iint_S (p_- - p_+) y \, dx \, dy = -2\rho c \iint_S \frac{\partial\varphi}{\partial x} y \, dx \, dy = 2\rho c \int_{-a}^{a} \Phi(y) y \, dy$$

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from which

$$M_x = \rho c \int_{-a}^{a} y\Gamma(y) dy \tag{3.18}$$

Expressing M_x in terms of $f(x,y)$ yields

$$M_x = -\frac{4}{3} \frac{\rho c a^2}{\pi^2} \iint_S \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta) \sin\gamma \cos\gamma \, d\gamma \, d\xi \, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos\gamma - 2a\eta \sin\gamma} \tag{3.19}$$

For the moment about the Oy axis,

$$M_y = -\iint_S (p_- - p_+) x \, dx \, dy = 2\rho c \iint_S x \frac{\partial\varphi}{\partial x} dx \, dy \tag{3.20}$$

Substituting the value $\partial\varphi/\partial x$ and integrating yield

$$M_y = -\frac{4\rho c}{\pi} \iint_S \left\{ 1 - \frac{a^2}{3\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^2\gamma \, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos\gamma - 2a\eta \sin\gamma} \right\} \sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta) d\xi \, d\eta \tag{3.21}$$

The following values are obtained for the coordinates of the center of pressure:

$$x_c = -\frac{M_y}{P}; \quad y_c = \frac{M_x}{P} \tag{3.22}$$

4. Examples

NACA comment: Errors in these examples are referred to and corrected in the paper "Steady Vibrations of Wing of Circular Plan Form".

The equations just obtained are presented again:

The velocity potential for $z > 0$ is determined by the equation

$$\varphi(x,y,z) = \frac{1}{2\pi}\iint_S \left\{ K(x,y,z,\xi,\eta) + \frac{1}{\pi^2\sqrt{2}} \times \right.$$

$$\left. \int_{+\infty}^{x}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - x^2 - y^2 - z^2 + R}\sqrt{a^2 - \xi^2 - \eta^2}\cos\gamma \, d\gamma \, dx}{(x^2 + y^2 + z^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \right\} f(\xi,\eta)\,d\xi\,d\eta \tag{4.1}$$

where

$$K(x,y,z,\xi,\eta) = \frac{2}{\pi r}\arc\tan\frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2 - z^2 + R}}{\sqrt{2}\,ar} \tag{4.2}$$

$$R = \sqrt{(a^2 - x^2 - y^2 - z^2)^2 + 4a^2z^2}; \quad r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2}$$

For the circulation distribution in the vortex strip formed behind the wing,

$$\Gamma(y) = -\frac{1}{\pi^3} \times$$

$$\iint_S \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\sqrt{a^2 - x^2 - y^2}\, f(\xi,\eta)\cos\gamma \, d\gamma \, dx \, d\xi \, d\eta}{(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{4.3}$$

$$= -\frac{1}{\pi^2}\iint_S \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\cos\gamma}{(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \left[\sqrt{\frac{a(1 \pm \sin\gamma)}{|a\sin\gamma - y|}} - 1\right] d\gamma \, d\xi \, d\eta$$

where the plus sign is taken for $y < a\sin\gamma$ and the minus sign for $y > a\sin\gamma$

The following expression gives the lift force:

$$P = \rho c \int_{-a}^{a} \Gamma(y)\,dy = -\frac{2\rho c a}{\pi^2}\iint_S \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\cos\gamma \, d\gamma \, d\xi \, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} \tag{4.4}$$

The usual expression for the induced resistance is

$$W = \frac{\rho}{4\pi}\int_{-a}^{a}\int_{-a}^{a} \Gamma(y)\frac{d\Gamma(y')}{dy'}\frac{1}{y - y'}\,dy\,dy' \tag{4.5}$$

The coordinates of the center of pressure are determined by the equations

$$x_c = -\frac{M_y}{P}; \quad y_c = \frac{M_x}{P} \tag{4.6}$$

where

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$$M_x = \rho c \int_{-a}^{a} y\Gamma(y)dy = -\frac{4}{3}\frac{\rho c a^2}{\pi^2}\iint_S \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\sin\gamma\cos\gamma\, d\gamma\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} \tag{4.7}$$

$$M_y = -\frac{4\rho c}{\pi}\iint_S \left\{1 - \frac{a^2}{3\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^2\gamma\, d\gamma}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma}\right\}\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)d\xi\, d\eta \tag{4.8}$$

If y is set equal to $-a\cos\theta$ and $\Gamma(y)$ is represented in the form of a trigonometric series,

$$\Gamma(y) = A_1 \sin\theta + A_2 \sin 2\theta + \ldots \qquad (0 < \theta < \pi) \tag{4.9}$$

P, W and M_x are directly expressed in terms of the coefficients of this series by the formulas

$$P = \frac{\pi\rho c a}{2} A_1; \quad W = \frac{1}{8}\pi\rho\sum_{n=1}^{\infty} nA_n^2; \quad M_x = -\frac{1}{4}\pi\rho c a^2 A_2 \tag{4.10}$$

Finally, the shape of the wing is determined by the equation

$$\zeta(x,y) = \frac{1}{c}\int_0^x f(x,y)dx - \frac{g(y)}{c}x + g_1(y) \tag{4.11}$$

where

$$g(y) = \frac{a}{2\pi^3} \times \iint_S \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, f(\xi,\eta)\cos\gamma\, d\gamma\, dx\, d\xi\, d\eta}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)(\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma)} \tag{4.12}$$

The examples are now considered.

1. First

$$f(x,y) = c\alpha$$

where α is a small constant.

Polar coordinates are used and the following integral computed:

$$\iint_S \frac{\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos \gamma - 2a\eta \sin \gamma}$$

$$= \int_0^a \int_0^{2\pi} \frac{\sqrt{a^2 - \rho^2}\, \rho\, d\vartheta\, d\rho}{a^2 + \rho^2 - 2a\rho \cos(\vartheta - \gamma)} = \int_0^a \frac{2\pi\rho\, d\rho}{\sqrt{a^2 - \rho^2}} = 2\pi a \quad (4.13)$$

Substituting this value in equation (4.3) yields

$$\Gamma(y) = -\frac{2ac\alpha}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos \gamma \left[\sqrt{\frac{a(1 \pm \sin \gamma)}{|a \sin \gamma - y|}} - 1\right] d\gamma$$

If the integral is taken,

$$\Gamma(y) = \frac{c\alpha}{\pi}\left\{-4a + 2\sqrt{2a(a - y)} + 2\sqrt{2a(a + y)} - (a + y) \log \frac{\sqrt{2a} - \sqrt{a - y}}{\sqrt{2a} + \sqrt{a - y}} - (a - y) \log \frac{\sqrt{2a} - \sqrt{a + y}}{\sqrt{2a} + \sqrt{a + y}}\right\} \quad (4.14)$$

Setting $y = -a \cos \theta$ and expanding $\Gamma(-a \cos \theta)$ in a trigonometric sine series in the interval $0 < \theta < \pi$ give after simple computations

$$\Gamma(-a \cos \theta) = \frac{ac\alpha}{\pi}\left\{-4 + 4 \cos \frac{\theta}{2} + 4 \sin \frac{\theta}{2} - (1 - \cos \theta) \log \frac{1 - \cos \frac{\theta}{2}}{1 + \cos \frac{\theta}{2}} - (1 + \cos \theta) \log \frac{1 - \sin \frac{\theta}{2}}{1 + \sin \frac{\theta}{2}}\right\}$$

$$= A_1 \sin \theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \ldots \quad (0 \leq \theta \leq \pi) \quad (4.15)$$

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where

$$A_1 = \frac{16ac\alpha}{\pi^2}\ ;\ A_{2k+1} = -\frac{4ac\alpha}{\pi^2 k(k+1)(2k+1)}\left(\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{4k+1}\right)$$

$$(k = 1, 2, \ldots) \tag{4.16}$$

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so that

$$A_3 = -\frac{16ac\alpha}{45\pi^2}\ ;\ A_5 = -\frac{496ac\alpha}{4725\pi^2}, \ldots$$

The distribution of the circulation obtained is very near that of an elliptical distribution.

The lift force and the induced drag are obtained by application of equations (4.10).

$$\begin{aligned} P &= \frac{\pi}{2}\rho ca A_1 = \frac{8}{\pi}\rho a^2 c^2 \alpha = 2.5465\ \rho a^2 c^2 \alpha \\ W &= \frac{1}{8}\pi\rho\left({A_1}^2 + 3{A_3}^2 + \ldots\right) = 1.034\ \rho a^2 c^2 \alpha^2 \end{aligned} \tag{4.17}$$

In order to determine the position of the center of pressure, M_y must be computed by equation (4.8).

Equation (4.13) gives

$$M_y = -\frac{4}{3}\rho c^2 a^3 \alpha\ ;\ x_c = -\frac{M_y}{P} = \frac{\pi}{6}a \tag{4.18}$$

The distance from the center of pressure, which evidently lies on the Ox axis, to the leading edge of the wing thus constitutes about 0.238 of the diameter of the wing.

In order to determine the shape of the wing corresponding to the assumed function, it is necessary to form the function $g(y)$ by equation (4.12). If equation (4.13) is considered,

$$g(y) = \frac{a^2 c\alpha}{\pi^2}\int_{\infty}^{\sqrt{a^2-y^2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos\gamma\, d\gamma\, dx}{\sqrt{x^2+y^2-a^2}\left(x^2+y^2+a^2-2ax\cos\gamma-2ay\sin\gamma\right)} \tag{4.19}$$

The computation shows that for $x > \sqrt{a^2 - y^2}$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos \gamma \, d\gamma}{x^2 + y^2 + a^2 - 2ax \cos \gamma - 2ay \sin \gamma} = - \frac{\pi x}{2a(x^2 + y^2)} +$$

$$\frac{y}{2a(x^2 + y^2)} \log \frac{x^2 + (y - a)^2}{x^2 + (y + a)^2} +$$

$$\frac{x(a^2 + x^2 + y^2)}{a(x^2 + y^2)(x^2 + y^2 - a^2)} \operatorname{arc\,tan} \frac{x^2 + y^2 - a^2}{2ax} \tag{4.20}$$

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If

$$y = - a \cos \theta ; \quad \sqrt{a^2 - y^2} = a \sin \theta$$

$$H_0(\theta) = \frac{\pi^2}{\alpha c} \sin \theta g(-a \cos \theta) \tag{4.21}$$

for $0 < \theta < \pi$

$$H_0(\theta) = \int_{+\infty}^{\sin \theta} \frac{\sin \theta}{\sqrt{t^2 - \sin^2\theta}} \left\{ - \frac{\pi t}{2(t^2 + \cos^2\theta)} - \frac{\cos \theta}{2(t^2 + \cos^2\theta)} \log \frac{t^2 + 4 \cos^4 \frac{\theta}{2}}{t^2 + 4 \sin^4 \frac{\theta}{2}} + \right.$$

$$\left. \frac{t(t^2 + 1 + \cos^2\theta)}{(t^2 + \cos^2\theta)(t^2 - \sin^2\theta)} \operatorname{arc\,tan} \frac{t^2 - \sin^2\theta}{2t} \right\} dt$$

Computation of this integral gives

$$H_0(\theta) = \left(\frac{\pi^2}{2} - 2 \int_0^1 \frac{\operatorname{arc\,tan} y}{\sqrt{1 - y^2}} dy \right) \sin \theta +$$

$$\frac{1}{8} \sin \theta \left(\log \frac{1 + \sin \frac{\theta}{2}}{1 - \sin \frac{\theta}{2}} \right)^2 + \frac{1}{8} \sin \theta \left(\log \frac{1 + \cos \frac{\theta}{2}}{1 - \cos \frac{\theta}{2}} \right)^2 +$$

$$\cos \frac{\theta}{2} \log \frac{1 - \sin \frac{\theta}{2}}{1 + \sin \frac{\theta}{2}} + \sin \frac{\theta}{2} \log \frac{1 - \cos \frac{\theta}{2}}{1 + \cos \frac{\theta}{2}} \tag{4.22}$$

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The shape of the wing is thus determined by the equation

$$\zeta(x,y) = \alpha x\left[1 - \frac{g(y)}{\alpha c}\right] = \alpha x\left\{\frac{1}{2} + \frac{2}{\pi^2}\int_0^1 \frac{\arctan y}{\sqrt{1 - y^2}}\,dy - \frac{1}{8\pi^2}\left(\log\frac{\sqrt{2a} + \sqrt{a + y}}{\sqrt{2a} - \sqrt{a + y}}\right)^2 - \frac{1}{8\pi^2}\left(\log\frac{\sqrt{2a} + \sqrt{a - y}}{\sqrt{2a} - \sqrt{a - y}}\right)^2 - \frac{\sqrt{2a}}{2\pi^2\sqrt{a + y}}\log\frac{\sqrt{2a} - \sqrt{a + y}}{\sqrt{2a} + \sqrt{a + y}} - \frac{\sqrt{2a}}{2\pi^2\sqrt{a - y}}\log\frac{\sqrt{2a} - \sqrt{a - y}}{\sqrt{2a} + \sqrt{a - y}}\right\} \qquad (4.23)$$

This wing differs little from a plane wing inclined to the xy-plane by a small angle α and may be obtained from such a plane wing by twisting. The values of the function $\zeta(x,y)$ for the mean value $y = 0$ and for the values $y = \pm a/2$ are

$$\zeta(x,0) = \alpha x\left[\frac{1}{2} + \frac{2}{\pi^2}\int_0^1 \frac{\arctan y}{\sqrt{1 - y^2}}\,dy - \frac{1}{\pi^2}\log^2(\sqrt{2} + 1) + \frac{2\sqrt{2}}{\pi^2}\log(\sqrt{2} + 1)\right] = 0.8452\ \alpha x$$

$$\zeta\left(x, \pm\frac{a}{2}\right) = \alpha x\left[\frac{1}{2} + \frac{2}{\pi^2}\int_0^1 \frac{\arctan y}{\sqrt{1 - y^2}}\,dy - \frac{1}{2\pi^2}\log^2(2 + \sqrt{3}) - \frac{1}{8\pi^2}\log^2 3 + \frac{2}{\pi^2\sqrt{3}}\log(2 + \sqrt{3}) + \frac{1}{\pi^2}\log 3\right] = 0.8335\ \alpha x$$

It is of interest to consider what results for the obtained wing are given by the usual theory. The circulation obtained by this theory is denoted by $\Gamma_0(y)$; if the expansion of this circulation in a trigonometric series is

$$\Gamma_0(-a\cos\theta) = B_1\sin\theta + B_2\sin 2\theta + \ldots \qquad (0 < \theta < \pi) \qquad (4.24)$$

then the usual theory gives an equation for determining the coefficients B_n, which in the case considered reduces to the form

$$\sum_{n=1}^{\infty} B_n \sin n\theta = 2\pi ca \sin\theta \left\{\alpha - \frac{g(-a\cos\theta)}{c} - \frac{1}{4ca}\sum_{n=1}^{\infty} nB_n \frac{\sin n\theta}{\sin\theta}\right\} \tag{4.25}$$

Equation (4.21) yields

$$\sum_{n=1}^{\infty} B_n\left(1 + \frac{\pi n}{2}\right) \sin n\theta = 2\pi ca\alpha \sin\theta - \frac{2a\alpha c}{\pi} H_0(\theta) \tag{4.26}$$

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Expansion of the function $H_0(\theta)$ into a trigonometric series is sufficient to determine the coefficients B_n. Despite the complicated form of the function $H_0(\theta)$, it can be expanded and in the interval $0 \leqslant \theta \leqslant \pi$

$$H_0(\theta) = \sin\theta\left(\frac{9\pi^2}{16} - 4 - 2\int_0^1 \frac{\text{arc tan } y}{\sqrt{1 - y^2}}\, dy\right) +$$

$$\sum_{k=1}^{\infty} \frac{\sin(2k + 1)\theta}{k(k + 1)}\left[-1 + \frac{1}{3} + \ldots + \frac{1}{4k + 1} - \frac{2(2k + 1)^2 + 1}{(4k + 1)(4k + 3)}\right] \tag{4.27}$$

that is,

$$H_0(\theta) = \sum_{k=0}^{\infty} \beta_{2k+1} \sin(2k + 1)\theta$$

where

$$\beta_1 = -0.1389;\quad \beta_3 = -0\ \text{[illegible]};\quad \beta_5 = -0.1213$$

$$\beta_7 = -0.0460;\quad \beta_9 = -0.0212,\ \ldots$$

Equation (4.26) shows that

$$B_1 = \frac{4\alpha ca(\pi^2 - \beta_1)}{\pi(\pi + 2)};\quad B_{2k} = 0;\quad B_{2k+1} = -\frac{4\alpha ca\beta_{2k+1}}{2 + \pi(2k + 1)}\quad (k = 1, 2, \ldots) \tag{4.28}$$

The numerical values of the first coefficients will be

$$B_1 = 2.4784\ \alpha ca;\quad B_3 = 0.0562\ \alpha ca;\quad B_5 = 0.0087\ \alpha ca$$

$$B_7 = 0.0024\ \alpha ca;\quad B_9 = 0.0009\ \alpha ca,\ \ldots$$

The following value is obtained for the lift force:

$$P_0 = \frac{1}{2}\pi\rho caB_1 = 3.8932\ \rho c^2a^2\alpha \tag{4.29}$$

exceeding the accurate value by 53 percent.

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For the induced drag,

$$W_0 = 2.416\ \rho a^2c^2\alpha^2 \tag{4.30}$$

with an error of 134 percent.

2. If α is assumed to be small, $f(x,y) = -2c\alpha x$ is taken.

The circulation $\Gamma(y)$ is computed. First the value of the following integral is found.

$$\iint_S \frac{\xi\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi\cos\gamma - 2a\eta\sin\gamma} = \frac{4}{3}\pi a^2\cos\gamma \tag{4.31}$$

Equation (4.3) gives

$$\Gamma(y) = \frac{8ca^2\alpha}{3\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos^2\gamma\left[\sqrt{\frac{a(1 \pm \sin\gamma)}{|a\sin\gamma - y|}} - 1\right] d\gamma$$

The computation of this integral leads to the very simple expression

$$\Gamma(y) = 2c\alpha(a^2 - y^2) \tag{4.32}$$

Thus in the case considered, a parabolic distribution of the circulation was obtained. For this reason the computation of the forces can be easily carried out:

$$\left.\begin{aligned} P &= \rho c\int_{-a}^{a}\Gamma(y)dy = \frac{8}{3}\alpha\rho c^2a^3 = 2.667\ \alpha\rho c^2a^3 \\ W &= \frac{4}{\pi}\rho c^2\alpha^2a^4 = 1.2732\ \rho c^2a^4\alpha^2 \end{aligned}\right\} \tag{4.33}$$

Equation (4.31) is used in the computation of M_y by equation (4.8):

$$M_y = \frac{128}{27\pi} \rho c^2 a^4 \alpha = 1.509 \rho c^2 a^4 \alpha ; \quad x_c = -\frac{M_y}{P} = -\frac{16}{9\pi} a \tag{4.34}$$

In order to determine the shape of the wing it is necessary to compute the function $g(y)$; equation (4.12) yields

$$g(y) = -\frac{4\alpha c a^3}{3\pi^2} \int_{+\infty}^{\sqrt{a^2-y^2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^2\gamma \, d\gamma \, dx}{\sqrt{x^2 + y^2 - a^2}(x^2 + y^2 + a^2 - 2ax \cos\gamma - 2ay \sin\gamma)}$$

Setting

$$H_1(\theta) = -\frac{3\pi^2}{4\alpha c a} \sin\theta g(-a\cos\theta) \quad (0 \leq \theta \leq \pi) \tag{4.35}$$

and carrying out the integration with respect to γ yield

$$H_1(\theta) = \int_{\infty}^{\sin\theta} \frac{\sin\theta}{(t^2 + \cos^2\theta)^2 \sqrt{t^2 - \sin^2\theta}} \left\{ t(t^2 + \cos^2\theta) + \right.$$

$$\frac{\pi}{4}(\cos^2\theta - t^2)(t^2 + 1 + \cos^2\theta) - \frac{1}{2} t \cos\theta (t^2 + 1 + \cos^2\theta) \log \frac{t^2 + 4\cos^4\frac{\theta}{2}}{t^2 + 4\sin^4\frac{\theta}{2}} +$$

$$\left. \frac{2(t^2 + \cos^2\theta)^2 + (t^2 - \cos^2\theta)\left[1 + (t^2 + \cos^2\theta)^2\right]}{2(t^2 - \sin^2\theta)} \arc\tan \frac{t^2 - \sin^2\theta}{2t} \right\} dt$$

Integration yields

$$H_1(\theta) = \frac{3\pi}{2} \left\{ \sin\theta \left(1 - \sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right) + \frac{1}{12} \log \frac{\left(1 + \cos\frac{\theta}{2}\right)\left(1 + \sin\frac{\theta}{2}\right)}{\left(1 - \cos\frac{\theta}{2}\right)\left(1 - \sin\frac{\theta}{2}\right)} + \right.$$

$$\left. \sin\theta\cos\theta \left[\log\tan\frac{\theta}{2} + \frac{1}{4}\log\frac{\left(1 + \cos\frac{\theta}{2}\right)\left(1 - \sin\frac{\theta}{2}\right)}{\left(1 - \cos\frac{\theta}{2}\right)\left(1 + \sin\frac{\theta}{2}\right)}\right] \right\} \tag{4.36}$$

In equation (4.11), the following is taken:

$$g_1(y) = \alpha(a^2 - y^2)$$

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Then for the function $\zeta(x,y)$, which determines the shape of the wing, the following expression is obtained:

$$\zeta(x,y) = \alpha(a^2 - x^2 - y^2) +$$

$$\frac{2\alpha a x}{\pi}\left\{1 - \sqrt{\frac{a+y}{2a}} - \sqrt{\frac{a-y}{2a}} + \frac{1}{12}\log\frac{(\sqrt{2a} + \sqrt{a-y})(\sqrt{2a} + \sqrt{a+y})}{(\sqrt{2a} - \sqrt{a-y})(\sqrt{2a} - \sqrt{a+y})} - \right.$$

$$\left. \frac{y}{a}\log\sqrt{\frac{a+y}{a-y}} - \frac{y}{4a}\log\frac{(\sqrt{2a} + \sqrt{a-y})(\sqrt{2a} - \sqrt{a+y})}{(\sqrt{2a} - \sqrt{a-y})(\sqrt{2a} + \sqrt{a+y})}\right\} \quad (4.37)$$

This wing is thus obtained as a deformation of the wing:

$$\zeta(x,y) = \alpha(a^2 - x^2 - y^2)$$

which for small α differs little from a segment of a sphere.

In particular, for $y = 0$,

$$\zeta(x,0) = \alpha(a^2 - x^2) + \frac{2\alpha a x}{\pi}\left[1 - \sqrt{2} + \frac{1}{3}\log(\sqrt{2} + 1)\right] = \alpha(a^2 - x^2 - 0.0767ax)$$

In order to apply the general theory to the obtained wing $H_1(\theta)$ is expanded into a trigonometric series:

$$H_1(\theta) = \left(\pi - \frac{17}{3} - \int_0^{\frac{\pi}{2}} \log\tan\frac{x}{2}\,dx\right)\sin\theta +$$

$$\sum_{k=1}^{\infty}\frac{\sin(2k+1)\theta}{4k(k+1)(2k-1)(2k+3)}\left[-12\pi k(k+1) + \right.$$

$$\left. 2(16k^2 + 16k - 3)\left(1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{1}{4k+1}\right) + 6(2k+1)\right]$$

$$= \sum_{k=0}^{\infty}\gamma_{2k+1}\sin(2k+1)\theta \quad (4.38)$$

where

$$\gamma_1 = -0.6931\;;\;\gamma_3 = -0.1783\;;\;\gamma_5 = -0.0812$$

$$\gamma_7 = -0.0463\;;\;\gamma_9 = -0.0300,\ \ldots$$

For the case considered, the usual theory gives for the determination of the circulation

$$\Gamma_0(-a \cos \theta) = B_1 \sin \theta + B_2 \sin 2\theta + \ldots \qquad (0 \leqslant \theta \leqslant \pi)$$

the equation

$$\sum_{n=1}^{\infty} B_n \sin n\theta = 2\pi ca \sin \theta \left\{ \alpha a \sin \theta - \frac{g(-a \cos \theta)}{c} - \frac{1}{4ca} \sum_{n=1}^{\infty} nB_n \frac{\sin n\theta}{\sin \theta} \right\} \tag{4.39}$$

Equation (4.35) and

$$\sin^2\theta = -\frac{8}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k + 1)\theta}{(2k - 1)(2k + 1)(2k + 3)} \qquad (0 \leqslant \theta \leqslant \pi) \tag{4.40}$$

give from equation (4.39) the equation

$$\sum_{n=1}^{\infty} B_n\left(1 + \frac{\pi n}{2}\right) = \sum_{k=0}^{\infty} \left[\frac{8\alpha a^2 c}{3\pi} \gamma_{2k+1} - \frac{16\alpha a^2 c}{(2k - 1)(2k + 1)(2k + 3)} \right] \sin(2k + 1)\theta \tag{4.41}$$

from which without difficulty B_n is obtained, in particular

$$B_{2k} = 0 \ ; \ B_1 = 1.8457 \ \alpha ca^2 \ ; \ B_3 = -0.2132 \ \alpha ca^2$$

$$B_5 = -0.0250 \ \alpha ca^2 \ ; \ B_7 = -0.0075 \ \alpha ca^2 \ ; \ B_9 = -0.0032 \ \alpha ca^2, \ \ldots$$

The following value is obtained for the lift force:

$$P = \frac{1}{2} \pi \rho c a B_1 = 2.899 \ \alpha c^2 a^3 \rho \tag{4.42}$$

exceeding the accurate value by 8.7 percent.

The induced drag

$$W = 1.3927 \ \rho \alpha^2 c^2 a^4 \tag{4.43}$$

exceeds the accurate value by 9.4 percent.

3. In order to give an example of a nonsymmetrical wing,

$$f(x,y) = \alpha c y$$

In this case it is first necessary to compute the integral

$$\iint_S \frac{\eta\sqrt{a^2 - \xi^2 - \eta^2}\, d\xi\, d\eta}{\xi^2 + \eta^2 + a^2 - 2a\xi \cos\gamma - 2a\eta \sin\gamma} = \frac{4}{3}\pi a^2 \sin\gamma \quad (4.44)$$

On account of equation (4.3),

$$\Gamma(y) = -\frac{4\alpha c a^2}{3\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin\gamma \cos\gamma \left[\sqrt{\frac{a(1 \pm \sin\gamma)}{|a \sin\gamma - y|}} - 1\right] d\gamma$$

After computing the integral,

$$\Gamma(y) = \frac{\alpha c}{\pi}\left[(a + y)\sqrt{2a(a - y)} - (a - y)\sqrt{2a(a + y)} + \right.$$

$$\frac{1}{6}(a + y)(a - 3y) \log\frac{\sqrt{2a} - \sqrt{a - y}}{\sqrt{2a} + \sqrt{a - y}} -$$

$$\left.\frac{1}{6}(a - y)(a + 3y) \log\frac{\sqrt{2a} - \sqrt{a + y}}{\sqrt{2a} + \sqrt{a + y}}\right] \quad (4.45)$$

is obtained.

Assuming $y = -a \cos\theta$ and expanding in a trigonometric series give

$$\Gamma(-a \cos\theta) = \frac{\alpha c a^2}{\pi}\left[2(1 - \cos\theta)\cos\frac{\theta}{2} - 2(1 + \cos\theta)\sin\frac{\theta}{2} + \right.$$

$$\frac{1}{6}(1 - \cos\theta)(1 + 3\cos\theta) \log\frac{1 - \cos\frac{\theta}{2}}{1 + \cos\frac{\theta}{2}} -$$

$$\left.\frac{1}{6}(1 + \cos\theta)(1 - 3\cos\theta) \log\frac{1 - \sin\frac{\theta}{2}}{1 + \sin\frac{\theta}{2}}\right]$$

$$= A_2 \sin 2\theta + A_4 \sin 4\theta + \ldots \qquad (0 \leqslant \theta \leqslant \pi) \quad (4.46)$$

where

$$A_2 = -\frac{128\ \alpha c a^2}{27\pi^2}$$

$$A_{2k} = \frac{4\alpha c a^2}{\pi^2}\left[\frac{8k^2 + 1}{8k(k^2 - 1)(4k^2 - 1)}\left(1 + \frac{1}{3} + \ldots + \frac{1}{4k - 1}\right) - \frac{2k}{(k^2 - 1)(4k^2 - 1)}\right] \qquad (4.47)$$

$$(k = 2,3, \ldots)$$

so that

$$A_2 = -0.4803\ \alpha c a^2\ ;\quad A_4 = 0.00549\ \alpha c a^2$$

$$A_6 = 0.00234\ \alpha c a^2\ ;\quad A_8 = 0.00121\ \alpha c a^2$$

Evidently there is no lift force, whereas for the induced drag the following value is obtained:

$$W = 0.1813\ \rho\alpha^2 c^2 a^4 \qquad (4.48)$$

The moment of the forces about the Ox axis is:

$$M_x = -\frac{1}{4}\pi\rho c a^2 A_2 = 0.3772\ \rho\alpha^2 c^2 a^4 \qquad (4.49)$$

The moment of the forces about the Oy axis is computed with the aid of equation (4.8), where use must be made of the result (4.44), and it is found that

$$M_y = 0 \qquad (4.50)$$

The following function is now computed:

$$g(y) = \frac{2\rho^3\alpha c}{3\pi^2}\int_{\infty}^{\sqrt{a^2-y^2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{\sin\gamma\cos\gamma\, d\gamma\, dx}{\sqrt{x^2 + y^2 - a^2}\,(x^2 + y^2 + a^2 - 2ax\cos\gamma - 2ay\sin\gamma)}$$

Setting

$$H_2(\theta) = \frac{3\alpha^2}{2a\pi c}\sin 2\theta(a\cos\theta) \qquad (0 \leq \theta \leq \pi) \qquad (4.51)$$

and carrying out the integration with respect to γ give

$$g_2(\theta) = \int_0^{\sin\theta} \frac{\sin\theta}{(t^2+\cos^2\theta)^2\sqrt{t^2-\sin^2\theta}}\left\{-\cos\theta(t^2+\cos^2\theta) + \frac{\pi}{2}\, t\cos\theta(t^2+1+\cos^2\theta) + \frac{1}{4}(t^2+1+\cos^2\theta)(\cos^2\theta - t^2)\log\frac{t^2+4\cos^4\frac{\theta}{2}}{t^2+4\sin^4\frac{\theta}{2}} - \frac{t\cos\theta\left[1+(t^2+\cos^2\theta)^2\right]}{t^2-\sin^2\theta}\arctan\frac{t^2-\sin^2\theta}{2t}\right\}dt \tag{4.52}$$

$$= \sin\theta\cos\theta\left\{\frac{3}{2}\log^2(\sqrt{2}+1) + 3\int_0^1 \frac{\arctan y}{\sqrt{1-y^2}}\,dy - \frac{3\pi^2}{4}\right\} +$$

$$\frac{\sin\theta(1-3\cos\theta)}{16}\left(\log\frac{1+\cos\frac{\theta}{2}}{1-\cos\frac{\theta}{2}}\right)^2 - \frac{\sin\theta(1+3\cos\theta)}{16}\left(\log\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}\right)^2 +$$

$$\frac{1+3\cos\theta}{2}\sin\frac{\theta}{2}\log\frac{1+\cos\frac{\theta}{2}}{1-\cos\frac{\theta}{2}} - \frac{1-3\cos\theta}{2}\cos\frac{\theta}{2}\log\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}$$

Expansion in a trigonometric series gives

$$g_2(\theta) = \sin 2\theta\left[\frac{3}{4}\log^2(\sqrt{2}+1) + \frac{3}{2}\int_0^1 \frac{\arctan y}{\sqrt{1-y^2}}\,dy - \frac{9\pi^2}{16} + \frac{2159}{630}\right] -$$

$$\sum_{k=2}^{\infty} \frac{2(8k^2+1)}{(4k^2-1)(4k^2-4)}\left(1+\frac{1}{3}+\ldots+\frac{1}{4k-1} - \frac{12k^2}{8k^2+1}\right)\sin 2k\theta$$

$$= \sum_{k=1}^{\infty} \delta_{2k}\sin 2k\theta \tag{4.53}$$

where

$$\delta_2 = -0.27412 \; ; \; \delta_4 = -0.08127 \; ; \; \delta_6 = -0.05198 \; ; \; \delta_8 = -0.03641, \ldots$$

The usual theory for determining the circulation

$$\Gamma_0(-a \cos \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$$

gives the equation

$$\sum_{n=1}^{\infty} B_n\left(1 + \frac{\pi n}{2}\right)\sin n\theta = 2\pi ca \sin \theta \left\{-a\alpha \cos \theta - \frac{g(-a \cos \theta)}{c}\right\}$$

$$= -\pi ca^2\alpha \sin 2\theta - \frac{4\alpha ca^2}{3\pi} H_2(\theta) \tag{4.54}$$

from which without difficulty

$$B_{2k+1} = 0 \quad (k = 0,1,2, \ldots)$$

$$B_2 = -0.7304 \; \alpha ca^2 \; ; \; B_4 = 0.0047 \; \alpha ca^2 \; ; \; B_6 = 0.0021 \; \alpha ca^2 \; ; \; B_8 = 0.0011 \; \alpha ca^2, \ldots$$

The lift force is found equal to zero and the induced drag and moment of the forces about the Ox axis are

$$W = 0.4191 \; \rho\alpha^2c^2a^4 \; ; \; M_x = 0.5737 \; \rho\alpha^2c^2a^4 \tag{4.55}$$

The first gives an error of 131 percent, the second of 52 percent.

By a combination of the obtained solutions it would have been possible to obtain further examples. From the examples given it is clear that for the case of a circular wing considerable deviations are obtained between the usual and the exact theories.

Translated by S. Reiss
National Advisory Committee
for Aeronautics

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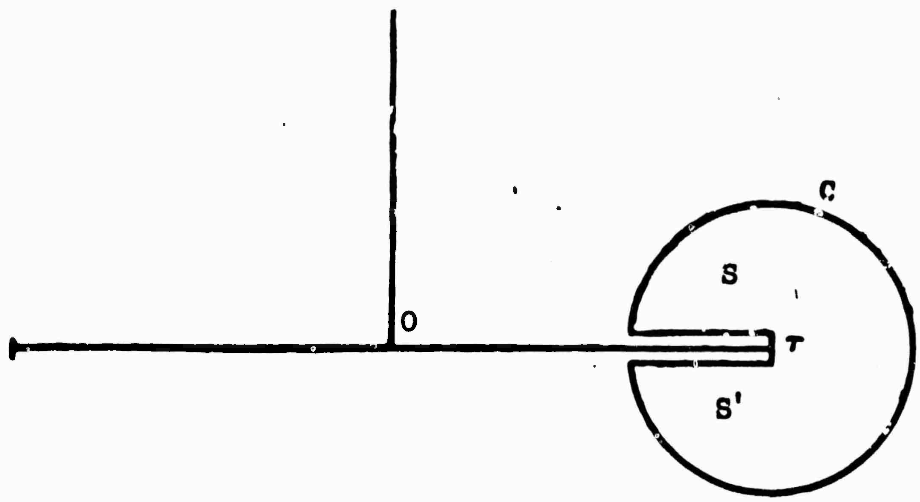

Figure 1.

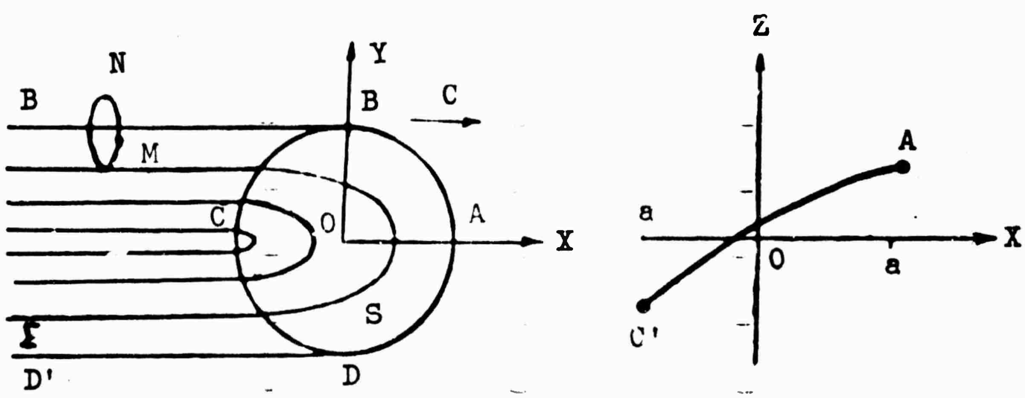

Figure 2.

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This paper treats the problem of determining the lift, moment, and induced drag of a thin wing of circular plan form in uniform incompressible flow on the basis of linearized theory. As contrasted to a similar paper by Kinner, in which the acceleration potential method was used, the present paper utilizes the concept of the velocity potential. Calculations of the

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lift and moment are presented for several deformed shapes. It is shown that considerable deviations exist between the strip theory analysis and the more exact theory. The lift, moment, and induced drag are also determined for a harmonically oscillatory circular plan form wing. As constrasted to a similar paper by Schade, in which the acceleration potential method was used, the present paper utilizes the concept of the velocity potential. Expressions for lift, moment, and induced drag are given and finally specialized to the case of a slowly oscillating circular wing.

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